Preacceleration in classical electrodynamics

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As is well known, the Lorentz-Dirac equation follows by means of a hypothesis of simplicity. When this hypothesis is ruled out, several additional terms can be added to the one that appears in the Lorentz-Dirac equation. We study the equation that considers the two terms explicitly shown by Dirac in his paper. It is shown, on general grounds, that the additional terms are not related to the radiation emitted by the electron; which is fully taken into account by the Larmor term of the Lorentz-Dirac equation. This result is explicitly verified by means of an exact solution of the enlarged Lorentz-Dirac equation that corresponds to a monoenergetic electron in circular orbit. Also it is shown that the additional term diminishes the effect of preacceleration in comparison with the one that comes from the Lorentz-Dirac equation.

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I. INTRODUCTION

The dynamics of the electron based on the concept of a point charge plays a fundamental role in classical as well as in quantum electrodynamics. At the classical level the point model has been of great importance in the description of the electron dynamics in particle accelerators; specially in connection with the spectrum of synchrotron radiation, which is correctly described by means of Schwinger's theoretical formula [1]. Besides, in quantum electrodynamics the point model of the electron has allowed one to calculate tiny effects, such as the shift of the energy levels in hydrogenlike atoms and corrections to the magnetic moment of the electron; calculations that agree with great accuracy with the experimental values. Moreover, high-energy experiments have not revealed any structure for the electron even at distances as small as 10^{-15} cm.

The fundamental equation of motion for a point electron in classical electrodynamics is the Lorentz-Dirac equation [2], which is

$$\dot{v}^{\mu}=(e/mc)F^{\mu\nu}v_{\nu}+\tau_0(\ddot{v}^{\mu}-\dot{v}^{\lambda}\dot{v}_{\lambda}v^{\mu}/c^2), \tag{1}$$

where e and m are the charge and mass of the electron, respectively, c is the velocity of light, v^{μ} is the four-velocity of the electron $\dot{z}^{\mu}$, and dots denote derivatives with respect to the proper time τ. Moreover, Greek indices take value 0,1,2,3, the metric is $(-1,1,1,1)$, the parameter τ_0 is given by

$$\tau_0=\frac{2e^2}{3mc^3}, \tag{2}$$

and Gaussian units are used throughout this paper. The first term on the right hand side of Eq. (1) is the acceleration due to the external field $F^{\mu\nu}$; while the second term is the contribution to the acceleration due to the electron's own field.

The Lorentz-Dirac equation (1) presents some pathological features, such us the existence of runaway solutions and acausal or preacceleration effects. To deal with these pathologies two methods have been worked out in the literature. In the first one the electron is still considered as a point particle, but alternative equations of motion to the Lorentz-Dirac equation have been proposed in order to avoid the existence of runaway solutions and preacceleration [3–9]. In particular, the Landau-Lifshitz equation [4] has gained renewed interest lately [10–12]. In the second method the idea of a point electron is abandoned in favor of an extended one [13–17], epecially by considering a model based in a uniformly charged spherical shell. This model leads to the Caldirola equation [18,19] in the relativistic case, equation that is free of runaway solutions and acausalities if the radius of the spherical shell is bigger than $2e^2/3mc^2$.

The models based on a point electron as well as that of a spherical shell cannot be considered as fully satisfactory. For example, the existence of divergent integrals in classical as well as quantum electrodynamics constitutes a clear indication that the electron cannot be represented by a point. The manipulation of quantities that take an infinitely large value is out of place in a definitive theoretical formulation.

The model where the electron is represented by a spherical shell has a rather hybrid structure, since the Caldirola equation determines, at least, in principle, a world line that describes the trajectory of a point. Physical motions such as rotations, deformations, or oscillations of the spherical shell, due to external fields or the fields of other charges, are not described by the world line. A fully relativistic formulation of an extended charge must be realized in terms of fields, including the electromagnetic field and also fields such as gravity and others that explain the stability of the electron. But a description in terms of fields necessarily involves a dynamical system with infinite degrees of freedom, and not just three, as the Caldirola equation does.

Unfortunately, a field theoretical formulation of an extended electron involves enormous mathematical complexities. Besides, the only structure dependent quantities of the electron that the experiment shows nowadays are its mass and charge. It is precisely due to this situation that simplified models, such as the point electron or the spherical shell, are of interest, since they may cast light into more realistic models. The domain of validity of classical electrodynamics is, of course, limited by quantum phenomena. Nevertheless, classical electrodynamics is still of interest in this context, since it must be obtained from the quantum formalism in the limit when Planck's constant tends to zero.

Even if the alternative equations of motion for a point charge do not have runaway solutions and acausal effects, they present, in general, other types of inconsistencies. For example, in the case of the Mo-Papas equation [5], Huschilt and Baylis [20] found evidence against its general validity. Besides, Comay [21] showed that the equations of Mo and Papas [5], Bonnor [6], and Herrera [7] are unphysical, because they do not satisfy the principle of conservation of the energy. Futhermore, we showed [22] that the Mo-Papas and Landau-Lifshitz equations admit as exact solutions, with appropriate external fields, the motion of a monoenergetic charge in circular orbit, as well as the motion of two identical charges that rotate at constant angular velocity at the opposite ends of a diameter. For these motions the rate of radiation can be determined by two independent methods. The first one uses the well-known Lienard-Wiechert fields of a point charge to calculate the energy flux across the surface on a sphere of an infinitely large radius that encloses the charges. This method allows one to obtain the correct rate of radiation, since it follows uniquely, without any ambiguities whatsoever, from the Maxwell equations. In the second method the rate of radiation is obtained starting from the Mo-Papas and Landau-Lifshitz equations by using the energy conservation law, the input energy due to the external fields, and the symmetries of the motions. It turns out that for the motion of one charge the rate of radiation obtained from the solution of the Mo-Papas equation coincides exactly with the one derived from the fields of a point charge for this motion. But in the case of the solution for the two charges, the solution of the Mo-Papas equation does not describe correctly the rate of radiation due to the interference of the fields of both charges. In the case of the Landau-Lifshitz equation both solutions lead to a rate of radiation that does not coincide with the one derived from the fields of a point charge. Of course, the difference between the rate of radiation calculated by the two procedures is, like the departure from causality in the case of the Lorentz-Dirac equation, very small.

As has been emphasized by Parrott [8] and Blanco [23], the above mentioned troubles that affect the equations of motion for a point charge do not mean that it is impossible to construct an equation of motion for a point charge consistent with basic principles such as the principle of causality and the principle of energy conservation, and that at the same time reproduces the rate of radiation obtained from the fields of a point charge. In this paper we will follow this conception. Although in addition to the runaway solutions and acausalities the Lorentz-Dirac equation also contains other defects as, for instance, instability problems [24,25] and lack of uniqueness [23,26–28], here we will focus mainly on the problem of preacceleration. Because of the well-known experimental limitations of classical electrodynamics, we will adopt consistency with basic principles as the guidance procedure, leaving aside experimental or practical aspects. The fundamental approach of Dirac [2] will be followed by considering the electron as a point from the start, but discarding Dirac's assumptions of simplicity. This opens the possibility that the preacceleration arises because the Lorentz-Dirac equation corresponds to a truncated version of an hypothetical "exact" equation which is free of acausal effects. Our conjecture is that in the "exact" equation the electron's own field would give rise to a power series in the parameter τ_0 of Eq. (2).

In this paper we present evidence in support of the above mentioned possibility about the existence of a classical equation of motion for a point charge without pathological features. Because of its great complexity, the problem will not be considered here in its full generality; but instead we will consider the enlarged Lorentz-Dirac equation that contains the two terms that appear explicitly in Dirac's paper. This equation is the following one:

$$\dot{v}^{\mu}=(e/mc)F^{\mu\nu}v_{\nu}+\tau_0(\ddot{v}^{\mu}-\dot{v}^{\lambda}\dot{v}_{\lambda}v^{\mu}/c^2)-\tau_0^4\{(4/c^4)\dot{v}^2 \times(\dot{v}\ddot{v})v^{\mu}+[(1/c^4)\dot{v}^4-(4/c^2)\ddot{v}^2-(4/c^2)(\dot{v}\dddot{v})]\dot{v}^{\mu} -(4/c^2)(\dot{v}\ddot{v})\ddot{v}^{\mu}\}. \tag{3}$$

The content of the paper is as follows. In Sec. II it is shown that the additional term proportional to τ_0^4 in Eq. (3) does not have any relation with the radiation emitted by the charge. As will be clearly established, the radiation emitted by the charge is fully taken into account by means of the nonlinear Larmor term that appears in the Lorentz-Dirac equation. The proof of this property is carried out in general grounds, and consequently, with independence of the enlarged equation under consideration, the radiation is exclusively described by the Larmor term of the Lorentz-Dirac equation.

In Sec. III, it is shown that Eq. (3), with appropriate external fields, allows as an exact solution the motion of a monoenergetic electron in a circular orbit. This solution reproduces exactly the same spectrum for synchrotron radiation as the Lorentz-Dirac equation does; that is, Schwinger's well-known formula.

In Sec. IV, Eq. (3) is applied to the motion of an electron along a straight line in the potential well, that is, in an external electric field that has a fixed value in a certain interval and vanishes identically outside it. Because of the high nonlinearity of Eq. (3), in this case an exact solution does not seem to be possible. However, by transforming Eq.(3) into an integro-differential equation, the solution can be obtained with any degree of accuracy. The relevant result is that the additional term proportional to τ_0^4 contains corrections to the preacceleration of order τ_0 that diminish the acausal effect.

Finally, Sec. V is devoted to some comments.

II. GEOMETRICAL GROUNDS

The purpose of this section is to clarify the physical meaning of the additional term proportional to τ_0^4 that figures in the enlarged Lorentz-Dirac equation (3), and more generally, the physical meaning of the almost arbitrary four-vector B_{μ} that appears in Dirac's paper [2]. To this end, two points of departure from Dirac's approach are of fundamental importance. The first consists in the use of only the physical fields of the electron, that is, the retarded fields. The second consists in following Rohrlich's approach based in the four-momentum associated with the electron's field [29], but considering now an arbitrary world line, and not just a charge in uniform motion.

In his paper Dirac computes the flow of the energy-momentum tensor across a thin world tube of radius ε that surrounds the electron world line $z_\mu(\tau)$. A point x_μ on this tube satisfies the following equations:

$$(x_\mu - z_\mu)(x^\mu - z^\mu) = \varepsilon^2, \qquad (x_\mu - z_\mu)v^\mu = 0. \tag{4}$$

For an arbitrary proper time τ, Eqs. (4) determine a two-dimensional sphere of radius ε centered at the point $z_\mu(\tau)$ and contained in the hyperplane orthogonal to the four-velocity $v_\mu(\tau)$. Therefore, an integral over the tube can be performed by integrating first over the two-dimensional sphere, followed by an integral over the proper time. Now, from the requirement of conservation of energy and momentum, Dirac concludes that the integral over the two-dimensional sphere must be a perfect differential of a four-vector B_μ. This four-vector B_μ is such that $\dot{B}_\mu$ is orthogonal to the four-velocity v_μ; but it is not determined by the conservation of energy and momentum. Thus a family of permissible B_μ exists, which in turns gives rise to a family of permissible equations of motion for a point electron. Dirac obtains the Lorentz-Dirac equation (1) by choosing the simplest B_μ, that is, the one proportional to the four-velocity v_μ. Dirac also presents in his paper another possible B_μ, which when added to the simplest one gives rise to the enlarged Lorentz-Dirac equation (3) that will be studied in the next two sections.

The fundamental quantity to be considered in this section is

$$P_\mu(\tau) = \frac{1}{c}\int_\Sigma T_{\mu\nu} d\Sigma^\nu, \tag{5}$$

where Σ is an arbitrary spacelike hypersurface that intercepts the electron world line at the point $z_\mu(\tau)$, and $T_{\mu\nu}$ is the usual energy-momentum tensor constructed with the retarded fields of the electron. Then, if $F_{\mu\nu}$ is the external field, the equation of motion of the electron would be

$$\frac{dP_\mu}{d\tau} = (e/c)F_{\mu\nu}v^\nu. \tag{6}$$

However, the energy-momentum tensor $T_{\mu\nu}$ has strong singularities at the point where the hypersurface Σ intercepts the electron world line, which implies that the integral (5) does not exist. In fact, $T_{\mu\nu}$ has three terms [30] that behave like ρ^{-4}, ρ^{-3}, and ρ^{-2}, where ρ is the invariant distance defined by $-v_\mu(x^\mu - z^\mu)/c$. In particular, the term that behaves as ρ^{-4} makes the integral in Eq. (5) not only ill defined, but divergent. This difficulty is, of course, directly associated with the concept of a point electron, and the standard solution is to perform a mass renormalization procedure.

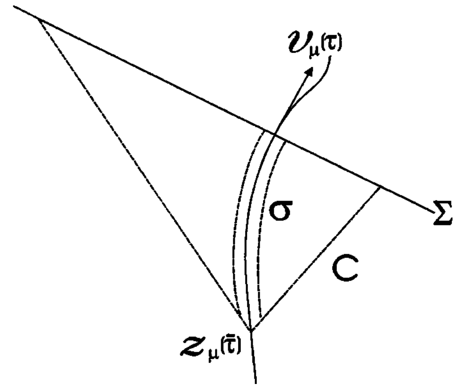

FIG. 1. The hyperplane Σ is orthogonal to the four-velocity v^μ, and σ is a Dirac tube that surrounds the electron world line. The light cone C is drawn from the straight part of the electron world line into the future.

In order to absorb the divergent part of P_μ by means of an electron mass renormalization procedure, some restrictions about the permissible hypersurface Σ that appears in Eq. (5) are necessary. Let us begin with Rohrlich's hyperplane Σ orthogonal to the electron world line [29]. Following also Rohrlich, let us isolate the point where the hyperplane cuts the electron world line by means of a two-dimensional sphere of radius ε contained in the hyperplane Σ and centered at the point of intersection. Instead of evaluating the integral (5) directly over the hyperplane, it is instructive to calculate it by an indirect way, with the help of Fig. 1 and the property

$$\partial^\nu T_{\mu\nu} = 0, \tag{7}$$

which is valid off the electron world line.

In Fig. 1 σ is the Dirac tube defined by means of Eqs. (4). For this reason the two-dimensional sphere at the upper end of the Dirac tube coincides exactly with the two-dimensional sphere in Rohrlich's hyperplane Σ. In other words, the Dirac tube σ and Rohrlich's hyperplane Σ fit perfectly well.

In what follows the electron world line is assumed to be an straight line in the remote past; which ensures that the total energy radiated by the electron is finite. The light cone C of Fig. 1 is drawn into the future from the point $z(\bar{\tau})$, located in the straight section of the electron world line. It is easy to see that in the straight section, but not in general, a two-dimensional sphere of the Dirac tube is such that all its points have a unique retarded point over the electron world line. Therefore, the lower end of the Dirac tube fits perfectly well with the light cone C.

In the limit when $\bar{\tau}$ goes to $-\infty$ and ε goes to zero, the integral over Rohrlich's hyperplane Σ can be transformed, with the help of Eq. (7), into an integral over Dirac's tube σ plus an integral over the light cone C, namely,

$$P_\mu(\tau) = \frac{1}{c}\int_\sigma T_{\mu\nu} d\sigma^\nu + \frac{1}{c}\int_C T_{\mu\nu} dC^\nu. \tag{8}$$

The integral over the Dirac tube has been evaluated elsewhere, [31,32] and the result is

$$\frac{1}{c}\int_{\sigma} T_{\mu\nu} d\sigma^{\nu} = \int_{-\infty}^{\tau} \left(\frac{e^2}{2\varepsilon c^2}\dot{v}_{\mu} - \frac{2e^2}{3c^3}\ddot{v}_{\mu} + \frac{2e^2}{3c^5}\dot{v}_{\lambda}\dot{v}^{\lambda}v^{\mu} \right) d\tau'. \tag{9}$$

The surface element of the light cone C is [33]: $dC^{\nu} = k^{\nu}\rho^2\, d\rho\, d\Omega$, where k^{ν} is the null ray $\{x^{\nu} - z^{\nu}(\overline{\tau})\}/\rho$, and $d\Omega$ is the solid angle element in the hyperplane orthogonal to $v_{\mu}(\overline{\tau})$. Besides $T_{\mu\nu}dC^{\nu} = (e^2/8\pi)k_{\mu}\rho^{-2}\, d\rho\, d\Omega$. Now, the integral over the light cone C is trivial because all the points of the lower two-dimensional sphere of the Dirac tube have $\rho = \varepsilon$. Moreover, in the limit when $\overline{\tau}$ goes to $-\infty$, the upper limit of ρ at Rohrlich's hyperplane is ∞. The result of the integral over the light cone C is equal to $(e^2/2\varepsilon c^2)v_{\mu}(-\infty)$. If, as usual, the bare mass four-momentum $m_0 v_{\mu}$ is added in order to carry out the renormalization process, the four-momentum P_{μ} associated with Rohrlich's hyperplane Σ is the following:

$$P_{\mu}(\tau) = m_0 v_{\mu}(\tau) + \int_{-\infty}^{\tau} \left(\frac{e^2}{2\varepsilon c^2}\dot{v}_{\mu} - \frac{2e^2}{3c^3}\ddot{v}_{\mu} + \frac{2e^2}{3c^5}\dot{v}_{\lambda}\dot{v}^{\lambda}v_{\mu} \right) d\tau' + \frac{e^2}{2\varepsilon c^2} v_{\mu}(-\infty), \tag{10}$$

from which, because of Eq. (6), the Lorentz-Dirac equation (1) follows.

The result (10) for the four-momentum (5) has been obtained for an arbitrary world line, except for the restriction of having a uniform motion in the remote past, and for an hyperplane Σ orthogonal to the electron world line at the point of intersection $z_{\mu}(\tau)$. In the particular case of a uniform motion for all proper time, that is, for $\dot{v}_{\mu}(\tau) \equiv 0$, Eq. (10) reduces to the well-known result of Rohrlich, since $v_{\mu}(-\infty) = v_{\mu}(\tau)$.

The above discussion shows that Rohrlich's well-known solution of the old problem about the 4/3 factor [29], is closely related with Dirac's derivation of the Lorentz-Dirac equation of motion (1) for a point electron. In this context, Rohrlich's choice of an hyperplane orthogonal to the four-velocity plays a crucial role. In fact, electron mass renormalization is possible if and only if the hyperplane Σ of Fig. 1 is orthogonal to the four-velocity [34].

The two first terms of the integrand of Eq. (10) are perfect differentials, and therefore after integration they represent quantities that depend only on the proper time τ associated with the point where Σ intersects the electron world line. The contribution at the lower limit of the integral cancels out with the term that contains $v_{\mu}(-\infty)$ in Eq. (10). On the contrary, the third term is not a perfect differential, and its integral contains information on the whole past history of the electron until the proper time τ.

In his study about the electromagnetic radiation, Rohrlich clearly identifies the third term in the integrand of Eq. (10) with the momentum four-vector rate at which radiation is leaving the charge [35]. The essentially different role of the third term with respect to the two first ones becomes transparent by means of Teitelboim's splitting of the energy-momentum tensor $T_{\mu\nu}$ into a bound, $T^{b}_{\mu\nu}$, and a radiation, $T^{r}_{\mu\nu}$, part [31,36].

$$T_{\mu\nu} = T^{b}_{\mu\nu} + T^{r}_{\mu\nu}. \tag{11}$$

The bound energy-momentum tensor $T^{b}_{\mu\nu}$ is defined by the terms of $T_{\mu\nu}$ that behave like ρ^{-4} and ρ^{-3}; whereas the radiation energy-momentum tensor $T^{r}_{\mu\nu}$ consists of the terms that behave like ρ^{-2}. The crucial property of the splitting (11) lies in that both parts are dynamically independent off the electron world line, since $T^{b}_{\mu\nu}$ as well $T^{r}_{\mu\nu}$ satisfy Eq. (7). Also, the splitting (11) allows to understand in a thorough way Rohrlich's local radiation criterion [35]. Thus $T^{r}_{\mu\nu}$ represents energy-momentum that detaches itself from the electron and leads an independent existence as soon as it is produced by the electron; whereas $T^{b}_{\mu\nu}$ represents energy momentum that is "tied" to the electron and is carried along with it.

The splitting (11) induces a natural splitting of the four-momentum (5) into a bound P^{b}_{μ} and a radiation P^{r}_{μ} part, each of which satisfies Eq. (8). In the case of P^{r}_{μ}, the integral over the light cone vanishes identically, and the integral over the Dirac tube gives rise to the nonlinear term that represents the radiation in the integrand of Eq. (10). However, in the case of P^{b}_{μ}, both the integral over the Dirac tube and the integral over the light cone are not vanishing. The integral over the Dirac tube gives rise to the perfect differentials that appear in the integrand of Eq. (10), while the integral over the light cone gives rise to the last term in the right hand side of Eq. (10).

The derivation of the Lorentz-Dirac equation (1) by means of the four-momentum P_{μ} given in Eq. (10) does not use the hypothesis of simplicity of Dirac. However, as will be clear below, Dirac's hypothesis of simplicity is implicitly contained in the geometry of Rohrlich's hyperplane. In general, the hypersurface Σ of Eq. (5) does not have to be an hyperplane; and then, because of the strong singularities of $T_{\mu\nu}$, different ways of isolating the point of intersection of the electron world line with the hypersurface Σ give rise to different P_{μ}, and consequently to different equations of motion.

Due to the different physical meaning of the radiation and bound energy momentum tensors $T^{r}_{\mu\nu}$ and $T^{b}_{\mu\nu}$, respectively, it is convenient to calculate the corresponding radiation and bound four-momentums P^{r}_{μ} and P^{b}_{μ} in a separate way. An instructive representation of them can be obtained with the help of Fig. 2. In this figure σ is an arbitrary two-dimensional surface contained in Σ that encloses the points of Σ that are in an immediate vicinity of the point $z_{\mu}(\tau)$ of intersection between Σ and the electron world line. C is the hypersurface constructed by means of null rays, where each ray is determined by a point of σ and its corresponding retarded point on the electron world line. The timelike tube Σ^{∞} surrounds the electron world line, and at the end the limit will be taken in which Σ^{∞} tends to spatial infinity.

Considering that the radiation tensor $T^{r}_{\mu\nu}$ satisfies Eq. (7), the radiation four-momentum P^{r}_{μ} can be written as an inte-

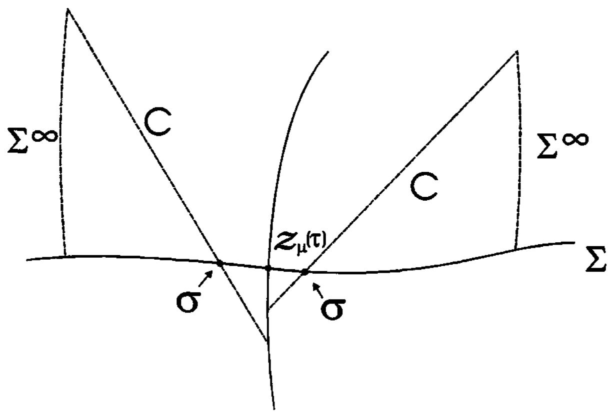

FIG. 2. Σ is an arbitrary spacelike hypersurface; σ is a two-dimensional surface contained in Σ and C is the three surface constructed by means of null rays.

gral over the three-surface C plus an integral over the tube Σ^∞ of Fig. 2. Due to the fact that the radiation energy momentum $T^r_{\mu\nu}$ is proportional to $k_\mu k_\nu$ and that the three-surface C is such that $k_\mu dC^\mu = 0$ [37], the integral over C vanishes identically no matter the form of Σ and σ. Therefore, only the integral over Σ^∞ survives in the case of P^r_μ. This integral has been evaluated elsewhere [31,34,36], and the result is the integral of the nonlinear Larmor term that appears in Eq. (10). Now, since P^r_μ is perfectly well defined, there is no room in it for the Dirac four-vector B_μ. In particular, then, this four-vector does not have any connection with the radiation emitted by the electron, which is fully taken into account by means of the Larmor term that appears in the Lorentz-Dirac equation (1). In the following section we will verify explicitly this property in the case of the exact solution of the enlarged Lorentz-Dirac Eq. (3) that describes a monoenergetic electron in circular orbit.

The bound four-momentum P^b_μ can be also written as the sum of an integral over the hypersurface C and an integral over the tube Σ^∞ of Fig. 2. But in this case, since the energy-momentum tensor $T^b_{\mu\nu}$ behaves at least as ρ^{-3} when ρ goes to infinity, the integral over Σ^∞ vanishes. Thus, only the integral over the hypersurface C survives and, as is evident, in the limit when the two-dimensional surface σ is shrunk into the electron world line P^b_μ, contrary to P^r_μ, depends only on the proper time τ. But now, due to the strong singularities of $T^b_{\mu\nu}$ at $\rho=0$, P^b_μ in contrast to P^r_μ is highly indeterminate. The condition for the renormalization of the mass requires that the surface Σ of Fig. 2 cuts the electron world line orthogonally, but this by no means determines P^b_μ [37]. Therefore, the four-vector B_μ of the Dirac paper is exclusively contained in the bound four-momentum P^b_μ [38]. Now, since P^b_μ depends only on the proper time τ, it gives rise to a perfect differential in the equation of motion (6), as the four-vector B_μ of Dirac does.

III. AN EXACT SOLUTION

The construction of the solution of the enlarged Lorentz-Dirac equation (3) corresponding to a monoenergetic electron in a circular orbit, needs the electrostatic field generated by an infinitely long solenoid of radius $\rho=b$, whose axis coincides with the Z axis, and which is fed with a time-dependent charge current density given by

$$\boldsymbol{J} = A t\,\delta(\rho-b)\hat{\boldsymbol{\varphi}}, \tag{12}$$

where A is a positive number, δ is the usual Dirac's delta function, ρ is the radial cylindrical coordinate, and $\hat{\boldsymbol{\varphi}}$ denotes the unit vector associated with the cylindrical coordinate φ. The charge density vanishes everywhere. These sources give rise to the following electric and magnetic fields, expressed in terms of their Cartesian components:

$$\begin{aligned} E_x &= (2\pi A\, b^2/c^2)\frac{y}{x^2+y^2},\\ E_y &= -(2\pi A\, b^2/c^2)\frac{x}{x^2+y^2},\\ E_z &= 0,\\ B_x &= B_y = B_z = 0, \end{aligned} \tag{13}$$

outside the solenoid, and

$$\begin{aligned} E_x &= (2\pi A/c^2)y,\\ E_y &= -(2\pi A/c^2)x,\\ E_z &= 0,\\ B_x &= B_y = 0, \quad B_z = (4\pi A t/c), \end{aligned} \tag{14}$$

inside the solenoid.

The electric field is tangent to the circles contained in planes parallel to the X-Y plane and centered at the Z axis. Moreover, the electric field has a fixed magnitude over each circle. It is immediate to see that the electromagnetic fields (13) and (14) satisfy the source free Maxwell equations, as well as the corresponding boundary conditions, at the solenoid surface $\rho=b$.

In what follows only the region outside the solenoid will be of interest. Besides, the time-independent electric field of Eq. (13), a time-independent homogeneous magnetic field that points along the Z axis will be assumed to exist outside the solenoid. As will be shown below, Eq. (3) with these external fields allows the solution corresponding to the motion of a monoenergetic electron in a circular orbit. Let us assume that the motion takes place in the X-Y plane in a circle of radius $a>b$ centered at the origin, that is

$$x^0 = c\,t, \quad x^1 = a\cos\omega t, \quad x^2 = a\sin\omega t, \quad x^3 = 0, \tag{15}$$

where ω is a time independent parameter. According to Eq. (13), the only nonzero Cartesian components of the external fields strengths $F^{\mu\nu}$ are

$$F^{01} = -F^{10} = E_x, \quad F^{02} = -F^{20} = E_y, \quad F^{12} = -F^{21} = B. \tag{16}$$

It is immediate to see that these external fields are such that the component $\mu=3$ of Eq. (3) is identically satisfied, without imposing any restriction on the parameters, if $v^3=\dot{x}^3\equiv 0$. Moreover, from Eq. (15) it follows that

$$v^0=c\gamma,\quad v^1=-c\beta\gamma\sin\omega t,\quad v^2=c\beta\gamma\cos\omega t, \tag{17}$$

where $\beta=a\,\omega/c$ and γ is the constant,

$$\gamma=\frac{dt}{d\tau}=(1-\beta^2)^{-1/2}. \tag{18}$$

If E denotes the magnitude of the time-independent electric field of Eq. (13) over the circle of radius a, then Eq. (3) for $\mu=1$ is

$$-a\omega^2\gamma^2\cos\omega t=(e\gamma E/m)\sin\omega t+(e\beta\gamma B/m)\cos\omega t + \tau_0\, a\,\omega^3\gamma^5\sin\omega t-\tau_0^4\beta^4 a\omega^6\gamma^{10}\cos\omega t, \tag{19}$$

whereas Eq. (3) for $\mu=2$ becomes

$$-a\omega^2\gamma^2\sin\omega t=-(e\gamma E/m)\cos\omega t+(e\beta\gamma B/m)\sin\omega t -\tau_0\, a\,\omega^3\gamma^5\cos\omega t-\tau_0^4\beta^4 a\omega^6\gamma^{10}\sin\omega t. \tag{20}$$

Here it is convenient to introduce the radial unit vector $\hat{\boldsymbol{\rho}}$ and the tangential unit vector $\hat{\boldsymbol{\varphi}}$ of the cylindrical coordinates, which in terms of the Cartesian unit vector $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are given by

$$\hat{\boldsymbol{\rho}}=\hat{\boldsymbol{i}}\cos\omega t+\hat{\boldsymbol{j}}\sin\omega t,$$

$$\hat{\boldsymbol{\varphi}}=-\hat{\boldsymbol{i}}\sin\omega t+\hat{\boldsymbol{j}}\cos\omega t,$$

which allows to write Eqs. (19) and (20) as

$$\{(e\gamma E/m)+\tau_0\, a\,\omega^3\gamma^5\}\hat{\boldsymbol{\varphi}} =[a\omega^2\gamma^2+(e\beta\gamma B/m)+\tau_0^4\beta^4 a\,\omega^6\gamma^{10}]\hat{\boldsymbol{\rho}}. \tag{21}$$

From which it follows that

$$E=-\frac{2e}{3a^2}\beta^3\gamma^4 \tag{22}$$

and

$$B=-\frac{mc\gamma\omega}{e}(1+\tau_0^4\omega^4\beta^4\gamma^8). \tag{23}$$

For $\mu=0$, Eq. (3) reproduces, once again, Eq. (22). The value of E that, according to Eq. (22) is needed to sustain the motion, can be easily obtained by choosing the appropriate value of the constant A that appears in the charge current density (12).

As is clear from Eq. (21), the term τ_0^4 of Eq. (3) appears exclusively in the radial component of this equation. The tangential component in Eq. (21) coincides exactly with the one obtained from the Lorentz-Dirac equation. The physical meaning of Eq. (22) becomes obvious when this equation is written in the form

$$e\boldsymbol{v}\cdot\boldsymbol{E}=\frac{2e^2c}{3a^2}\beta^4\gamma^4, \tag{24}$$

where $\boldsymbol{v}$ is the ordinary velocity $\boldsymbol{v}=v\hat{\boldsymbol{\varphi}}$. The left hand side of Eq. (24) corresponds to the power that the external electric field supplies to the electron. Therefore, because of energy conservation and the fact that the kinetic energy of the electron remains fixed, the right hand side must be the total rate of radiation that escapes to infinity. But the right hand side of Eq. (24) is exactly the total rate of radiation that follows from the nonlinear term of the Lorentz-Dirac equation [39]. This result constitutes a verification, for this particular motion, that the τ_0^4 term of Eq. (3) has no relation with the energy that is radiated away, in agreement with the analysis carried out in Sec. II. In comparison with the Lorentz-Dirac equation (1), the only effect of the τ_0^4 term of the enlarged Lorentz-Dirac equation (3) consists of a tiny change in the magnetic field from that required by the Lorentz-Dirac equation for the same motion.

IV. THE PREACCELERATION

Although the term proportional to τ_0^4 of the enlarged Lorentz-Dirac equation (3) does not affect the radiation, it has, nevertheless, an important influence on the phenomenon of preacceleration. In order to see this it is enough to consider the case of a motion along a straight line, which will be chosen as the X axis. The external electromagnetic field consists then in an electric field that has a nonvanishing component only along the X axis. Thus, the only nonzero components of the fields strengths $F^{\mu\nu}$ are

$$F^{01}=-F^{10}=E. \tag{25}$$

For this $F^{\mu\nu}$, the components $\mu=2$ and $\mu=3$ of Eq. (3) are identically satisfied with $v^2=v^3=0$. Now, as in the case of the Lorentz-Dirac equation [40], here it is also convenient to write the components v^0 and v^1 of the four-velocity in terms of the rapidity $w(\tau)$ as follows:

$$\begin{aligned} v^0&=c\cosh(w/c),\\ v^1&=c\sinh(w/c). \end{aligned} \tag{26}$$

The component with $\mu=1$ of Eq. (3) then becomes

$$\dot{w}-\tau_0\ddot{w}=f(\tau)+\frac{\tau_0^4}{c^2}(4\dot{w}^2\dddot{w}+8\dot{w}\ddot{w}^2-\dot{w}^5/c^2), \tag{27}$$

where $f(\tau)=eE(\tau)/m$. The component with $\mu=0$ of Eq. (3) reproduces once again Eq. (27). In contrast to the case of the Lorentz-Dirac equation, the introduction of the rapidity $w(\tau)$

by means of Eq. (26) does not linearize Eq. (3), because of the strong nonlinearities contained in the additional term proportional to τ_0^4. Fortunately, the effect of the nonlinear terms in Eq. (27) on the preacceleration can be studied without the necessity of constructing an exact solution of Eq. (27). Approximate solutions of Eq. (27) can be constructed with high accuracy by the method of successive approximations developed by Aguirregabiria [41]. But here it appears to be more suitable to transform Eq. (27) into an integro-differential equation. To this end, let us remember that the solution of the Lorentz-Dirac equation

$$\dot{w}-\tau_0\ddot{w}=f(\tau), \tag{28}$$

free of runaway behavior is given by

$$\dot{w}=\int_0^\infty e^{-s}f(\tau+\tau_0 s)ds.$$

This equation can be also written in the more convenient form,

$$\dot{w}=\frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0}f(\tau')d\tau'. \tag{29}$$

From Eq. (29) and Eq. (27) it follows that the solution $w(\tau)$ of Eq. (27) satisfies the following integrodifferential equation:

$$\dot{w}=\frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0}\left\{f(\tau')+\frac{\tau_0^4}{c^2}(4\dot{w}^2\dddot{w}+8\dot{w}\ddot{w}^2-\dot{w}^5/c^2)\right\}d\tau'. \tag{30}$$

If instead of the exact solution $w(\tau)$, the solution $w_1(\tau)$ of the Lorentz-Dirac equation (28) is introduced in the integrand of Eq. (30), then

$$\dot{w}_2=\frac{e^{\tau/\tau_0}}{\tau_0}\int_\tau^\infty e^{-\tau'/\tau_0}\left\{f(\tau')+\frac{\tau_0^4}{c^2}(4\dot{w}_1^2\dddot{w}_1+8\dot{w}_1\ddot{w}_1^2-\dot{w}_1^5/c^2)\right\}d\tau' \tag{31}$$

will be an approximate solution for the acceleration $\dot{w}$ of Eq. (27). The exact solution of Eq. (30) can be then obtained by means of successive iterations. Now, if $f(\tau)$ vanishes for large τ, then according to Eq. (29) $\dot{w}_1(\tau)$ also vanishes for large τ. This in turn implies that the $\dot{w}_2(\tau)$ of Eq. (31) also vanishes for large τ. This property is valid, of course, for any iteration. Therefore the exact solution of Eq. (27) constructed in this manner has an "acceleration" $\dot{w}(\tau)$ that also vanishes for large τ. In other words, this procedure ensures that the solution does not have a runaway behavior.

For the purpose of this paper it is enough to consider the correction to the Lorentz-Dirac equation contained in Eq. (31). Moreover, for definiteness the following pulse $f(\tau)$ will be considered:

$$f(\tau)=\begin{cases} f_0 & \text{for} \quad 0<\tau<\tau_1 \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

where f_0 is a constant. The corresponding solution of the Lorentz-Dirac equation (28) is given by

$$\dot{w}_1(\tau)=\begin{cases} f_0 e^{\tau/\tau_0}(1-e^{-\tau_1/\tau_0}) & (\tau<0) \\ f_0(1-e^{-(\tau_1-\tau)/\tau_0}) & (0<\tau<\tau_1) \\ 0 & (\tau_1<\tau). \end{cases} \tag{33}$$

Since the interest here is in the preacceleration, the $\dot{w}_2(\tau)$ of Eq. (31) will be considered only for negative values of the proper time τ. Moreover, only the leading contribution to the preacceleration due to the additional term proportional to τ_0^4 of Eq. (3) will be given. Despite that this term is proportional to τ_0^4, it gives rise to a contribution of order τ_0 to the preacceleration.

Because of the jumps that the function $f(\tau)$ in Eq. (32) has at the proper times $\tau=0$ and $\tau=\tau_1$, the integrand of Eq. (31) has delta functions at these proper times. The derivatives $\ddot{w}_1$ and $\dddot{w}_1$ that appear in the integrand of Eq. (31) can be easily obtained from Eq. (28) for any τ, namely,

$$\ddot{w}_1=-\frac{1}{\tau_0}\{-\dot{w}_1+f_0[\theta(\tau)-\theta(\tau-\tau_1)]\},$$

$$\dddot{w}_1=-\frac{1}{\tau_0^2}\{-\dot{w}_1+f_0[\theta(\tau)-\theta(\tau-\tau_1)]+\tau_0 f_0[\delta(\tau)-\delta(\tau-\tau_1)]\}, \tag{34}$$

where $\theta(\tau)$ is the step function and $\delta(\tau)$ is the usual Dirac delta function. From Eq. (33) it is easy to see that the term $-\tau_0^4\dot{w}_1^5/c^4$ that appears in the integrand of Eq. (31) gives rise to contributions of order τ_0^4 and τ_0^3 to the preacceleration. The term $8\tau_0^4\dot{w}_1\ddot{w}_1^2/c^2$ gives rise to a correction of order τ_0^2. The only contribution of order τ_0 comes from the term $4\tau_0^4\dot{w}_1^2\dddot{w}_1/c^2$ and it is the following:

$$\tau_0\left(\frac{4\tau_1 f_0^3}{c^2 e^{\tau_1/\tau_0}}\right)e^{\tau/\tau_0}.$$

Therefore, for $\tau<0$ the Lorentz-Dirac preacceleration that appears in Eq. (33) is changed to

$$f_0 e^{\tau/\tau_0}(1-e^{-\tau_1/\tau_0})\left\{1-\tau_0\frac{4\tau_1 f_0^2}{c^2(e^{\tau_1/\tau_0}-1)}+O(\tau_0^2)\right\}. \tag{35}$$

In other words, the term proportional to τ_0^4 that figures in the enlarged Lorentz-Dirac equation (3) diminishes the effect of preacceleration with respect to the one that appears in the

Lorentz-Dirac equation. This result suggests the possibility that the preacceleration may be, perhaps, suppressed in the enlarged Lorentz-Dirac equation that considers the complete series with all the powers of τ_0.

V. A FEW COMMENTS

We have discussed the preacceleration in Eq. (3) by considering the pulse (32) that changes abruptly at $\tau=0$ and $\tau=\tau_1$. However, we want to emphasize that the preacceleration present in the Lorentz-Dirac equation is not a result of forces that change too fast with time. In order to clarify this point, let us consider the motion of a charge $e>0$ along the positive x axis in an electrostatic field $\boldsymbol{E}(x)$ that has only one component along the positive x axis, and such that $\boldsymbol{E}(x)$ vanishes identically for $x<x_1$ and $x>x_2$. Then, by associating the proper time $\tau=0$ with the point $x=x_1$, Eq. (29) that is valid for continuous as well as discontinuous forces becomes

$$\dot{w}=\frac{e^{\tau/\tau_0}}{\tau_0}\int_0^{\tau_1} e^{-\tau'/\tau_0} f(\tau')\,d\tau' \tag{36}$$

for $\tau<0$, where $\tau_1>0$ is the proper time corresponding to the point $x=x_2$. Equation (36) clearly shows that $\dot{w}(\tau)$ is positive for $\tau<0$, irrespective of the smoothness properties of $f(\tau)$. Therefore, even if $f(\tau)$ starts to increase sufficiently smoothly at $x=x_1$, the preacceleration is anyway present. We have chosen the pulse (32) because it leads to the simple formula (33).

In the literature there are several derivations of the Lorentz-Dirac equation where the hypothesis of simplicity does not appear. However, in those derivations there are some assumptions that, in general, are not explicitly mentioned, and which play a role similar to the simplicity one. We cannot discuss here the different derivations of the Lorentz-Dirac equation, but in order to illustrate this point we will briefly comment on the derivation presented in Sec. II and on that of Barut [42].

If we calculate the four-momentum [5] with the help of the construction of Fig. 1, we obtain, because of Eq. (6), the Lorentz-Dirac equation (1) at once, without the need of the hypothesis of simplicity. But as we have already pointed out in Sec. II, the hypothesis of simplicity is implicitly contained in the very special choice of the hypersurfaces of Fig. 1, as becomes clear when we consider the hypersurfaces of Fig. 2.

Barut [42] has given a very nice derivation of the Lorentz-Dirac equation (a procedure that has been extended to other cases [43,44]), where at first sight an ingredient such as the hypothesis of simplicity does not arise. However, in this case the ingredient that plays a role analogous to the simplicity one consists in the procedure itself; since even if the electron field is considered as given by the Lienard-Wiechert formula, the field point and the retarded one are worked out as independent variables. Although this procedure is very nice and somewhat reasonable, it does not follow from basic principles.

The present approach to the preacceleration has been mainly motivated by the lack of a correct description of the radiation in several alternative equations of motion for a point charge [20–22]. In our approach the coincidence between the rate of radiation inferred from the equation of motion with the one obtained from the fields of a point charge is fully guaranteed, at least in the one charge case, since irrespective of the enlarged Lorentz-Dirac equation, the rate of radiation is given by the Larmor term. Besides, it is also reasonable to expect that any admissible B_μ will change the preacceleration contained in the Lorentz-Dirac equation. This expectation is based on the fact that each admissible B_μ, like the Schott term [31,36] that gives rise to the preacceleration in the Lorentz-Dirac equation, is associated with the bound part of the energy-momentum tensor, as was shown in Sec. II.

Our conjecture about the suppression of the preacceleration when the self-field of the electron is considered as a power series in the parameter τ_0, appears to be somewhat reasonable, not only because of the results of Sec. IV, but also because the Caldirola equation (free of preacceleration) can be written as a power series, involving the radius of the spherical shell, that resembles the power series in τ_0. Of course, these power series are not the same, because in the limit when the radius of the spherical shell goes to zero, the equation for a point charge that results is the Lorentz-Dirac equation, and not the enlarged one.

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